LOCALIZATION, UNIVERSAL PROPERTIES, AND HOMOTOPY THEORY

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- Localization in Algebra
- Localization in Category Theory
- Bousfield localization

1. The right way to think about localization in algebra

Localization is a **systematic way of adding multiplicative inverses to a ring**, i.e. given a commutative ring R with unity and a multiplicative subset $S \subset R$ (i.e. contains 1, closed under product), localization constructs a ring $S^{-1}R$ and a ring homomorphism $j : R \to S^{-1}R$ that takes elements in S to units in $S^{-1}R$. We want to do this in the best way possible, and we formalize that via a universal property, i.e. for any $f : R \to T$ taking S to units we have a unique g:

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d, "f"'] & S^{-1}R \arrow[dl, dashed, "g"] \\ T & \end{tikzcd}$$

Recall that $S^{-1}R$ is just $R \times S/\sim$ where (r, s) is really r/s and $r/s \sim r'/s'$ iff $t(rs' - sr') = 0$ for some t (i.e. fractions are reduced to lowest terms). The ring structure can be verified just as for $\mathbb{Q}$. The map j takes $r \mapsto r/1$, and given f you can set $g(r/s) = f(r)f(s)^{-1}$. Demonstrate commutativity of the triangle here. The universal property is saying that $S^{-1}R$ is the closest ring to R with the property that all $s \in S$ are units. A category theorist uses the universal property to define the object, then uses $R \times S/\sim$ as a construction to prove it exists. An algebraist might define the localization to be $R \times S/\sim$ and then prove the universal property as a corollary. It's a philosophical difference.

Examples:

- $\mathbb{Z}, \mathbb{Z} - \{0\}) \mapsto S^{-1}R = \mathbb{Q}$. More generally: $Frac(R)$
- $(\mathbb{Z}, \langle 2 \rangle) \mapsto \mathbb{Z}[\frac{1}{2}]$ the dyadic rationals.
- $(\mathbb{Z}, \mathbb{Z} - p\mathbb{Z}) \mapsto \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \not| b\}$
- $R = \mathbb{Q} \times \mathbb{Q}$, $S = (1, 0)$ has $S^{-1}R = \mathbb{Q} \times \{0\} \cong \mathbb{Q}$, smaller than R. Indeed, $R \to S^{-1}R$ is an injection iff S does not contain any zero divisors.

This is NOT the right definition to a category theorist (no operation, so "multiplicative inverses?"). Better: **systematic way of formally inverting maps**. We can't do this for all maps, but we can do it for maps of the form $\mu_s : R \to R$ which take $r \mapsto s \cdot r$ for an element s.

Definition 1. *Given $s \in R$ a localization is a ring R_* containing s, such that*

(1) $\mu_s : R_* \to R_*$ is an isomorphism

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(2) *R_* is universal with respect to this property, i.e. there is a map $i : R \to R_*$ and any time a map $g : R \to T$ takes μ_s to an isomorphism,*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R_* \arrow[d, dotted] \\ & T \end{tikzcd}$$

We prove these two notions of localization are the same, i.e. produce isomorphic rings. This is an example of finding the right proof in algebra to generalize to category theory. It's diagrammatic.

Proposition 2. $R_* \cong s^{-1}R$

Proof. We need maps $g : R_* \to s^{-1}R$ and $h : s^{-1}R \to R_*$ with $hg = id$ and $gh = id$. We'll get them by proving first $R \to s^{-1}R$ satisfies the universal property for R_* and second that $R \to R_*$ satisfies the universal property for $s^{-1}R$

Certainly $s \in s^{-1}R$ as $(s, 1)$. Also, μ_s is an isomorphism with inverse $\mu_{s^{-1}}$. So by the universal property of R_*, the map $j : R \to s^{-1}R$ gives $g : R_* \to s^{-1}R$ s.t. $g \circ i = j$

Next, the element s has an inverse in R_* because it's $\mu_s^{-1}(1)$ as $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of $s^{-1}R$ gives $h : s^{-1}R \to R_*$ and

$$\begin{tikzcd} & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\ R_* \arrow[r, "g"'] & s^{-1}R \arrow[r, "h"'] & R_* \end{tikzcd}$$

The bottom is the identity because the two triangles are the same. So $h \circ g = id_{R_*}$. Same idea gets $g \circ h$. □

Punchline: Localization should be thought of as inverting maps

Second punchline: Universal properties are cool

2. Categories

A category $\mathcal{C}$ is a **class of objects** linked by arrows which preserve the structure of the objects. Formally, between any two objects X, Y we have a **set of morphisms** $\mathcal{C}(X, Y)$ such that there is always $1_X \in \mathcal{C}(X, X)$ and if $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ then $g \circ f \in \mathcal{C}(X, Z)$. The composition is associative and unital ($1_Y \circ f = f = f \circ 1_X$).

An isomorphism is a morphism $f : X \to Y$ such that there is some $h : Y \to X$ with $f \circ h = 1_Y$ and $h \circ f = 1_X$.

The notion of category provides a fundamental and abstract way to describe mathematical entities and their relationships. Virtually every branch of modern mathematics can be described in terms of categories. Thus, if we can phrase a concept in terms of categories, it will have versions in most fields of mathematics. Because of the power of the results, proofs can be technical, but life can be made much easier by finding the right definitions and proofs classically first, as we did above.

Examples:

- Set, the category of sets and set functions. Isomorphisms are bijections
- Grp, the category of groups and group homomorphisms. Isomorphisms.

- Ab, the category of abelian groups and group homomorphisms. Isomorphisms.
- Ring, the category of rings and ring homomorphisms. Similarly, CRing.
- R-Mod, the category of R-modules and module homomorphisms.
- Ch(R), the category of chain complexes over R (descending sequence A_n with $d^2 = 0$) with chain maps (f_n) with $f \circ \partial = \partial \circ f$. Isomorphisms are levelwise.

$$
\begin{tikzcd}
\cdots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"'] & \cdots \\
\cdots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \cdots
\end{tikzcd}
$$

- Top, the category of topological spaces and continuous maps. Homeomorphisms
- HoTop, the category of topological spaces and homotopy classes of maps. Isomorphisms are now homotopy equivalences.
- Ho(Ch(R)), the category of chain complexes with chain homotopy classes of maps.

$$
\begin{tikzcd}
\cdots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[dl, "h^{n-1}"'] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[dl, "h^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[dl, "h^{n+1}"'] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] & \cdots \arrow[dl, "h^{n+2}"'] \\
\cdots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \cdots
\end{tikzcd}
$$

Compare this notion to that of a quasi-isomorphism, i.e. a map $f : A_\bullet \to B_\bullet$ which that $H_*(f)$ is an isomorphism of graded abelian groups. This gives a different homotopy category (called the derived category of R) because not every quasi-isomorphism is a chain homotopy equivalence.

The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to functors $F : \mathcal{C} \to \mathcal{D}$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$ (i.e. they preserve structure). Formally, $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Examples:

- Forgetful functor: Grp $\to$ Set
- Free functor: Set $\to$ Grp. Or free abelian functor: Set $\to$ Ab
- Abelianization functor: Grp $\to$ Ab
- Inclusion: R-Mod $\to$ Ch(R), including the module in at level 0.

3. Localization for Categories

Thinking of localization as "formally inverting maps" then we want to pick a set W of morphisms and create a universal functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in W$. Universal means if there is $\mathcal{C} \to \mathcal{D}$ taking W to isomorphisms then we have $\mathcal{C}[W^{-1}] \to \mathcal{D}$ making the triangle commute.

Example (to remind ourselves that this is still a topology talk): If $\mathcal{C}$ is Top, and we want to study it "up to homotopy" (i.e. when X h.e. Y we say they are isomorphic), then we get the homotopy category. This functor zooms in on homotopy theoretic information by formally setting the homotopy equivalences to be isomorphisms. In $HoTop$ some spaces are declared to be "the same" up to isomorphism even if they are not homeomorphic. This process is universal because we added the smallest numbers of isomorphisms possible.

It is not true that for every choice of category and class of maps $\mathcal{C}, W$ there is a localized category $\mathcal{C}[W^{-1}]$. If we try to construct the category $\mathcal{C}[W^{-1}]$ in the only reasonable way, we see that the universal property forces it to have the same class of objects as $\mathcal{C}$. The morphisms are trickier. Given $f : X \to Y$ in W and $g : X \to Z$, we get $g \circ f^{-1} : Y \to Z$, i.e. we have to generate new morphisms based on the inverses I added. You can get there by any **zig-zag (DRAW IT)**, so you want to define $\mathcal{C}[T^{-1}](X, Y) = \{X \leftarrow \bullet \to \bullet \cdots \bullet \to Y\}/\sim$ where this relation at least allows us to add in pairs of identities or compose two when it's allowed. PROBLEM: the collection of zigzags $X \leftarrow \bullet \to \bullet \cdots \bullet \to Y$ is not a set; even just in the category Set you have a proper class worth of choices.

Attempting to get around these set-theoretic issues **leads you to model categories** (invented by Quillen in 1967). So you make some assumptions so that $\mathcal{C}$ behaves more like Top and W behaves like the (weak) homotopy equivalences. The idea is you have a special class of maps $\mathcal{W}$ called the weak equivalences, and these generalize the homotopy equivalences above. Quillen was able to define a very general notion of homotopy, via cylinder objects and path objects. To prove it's an equivalence relation is hard.

Quillen's clever observation was that in the examples of interest you can always replace your object by a nicer one which is homotopy equivalent. For spaces this can be CW approximation. For R-Mod and Ch(R) it can be **projective and injective resolution**. These are all cases where you can build more complicated object from simpler ones. Borrowing terminology from topology, we call a map a **cofibration** if we build it via colimits, wedges, and retracts of spheres and disks. In algebra this means building it from chains of free modules, so applying this process to an object $M_\bullet$ gives a projective resolution $P_\bullet \to M_\bullet$ called the cofibrant replacement of $M_\bullet$ (since the domain is cofibrant and the map is a quasi-isomorphism).

Dually, we can build an injective resolution for $M_\bullet$ by mapping to an injective object then taking the cokernel and mapping that to an injective and continuing in this way. Again we are building a chain complex up inductively from pieces we understand, but the pieces are dual to the above and now we get a map $M_\bullet \to I_\bullet$. Objects built in this way are called fibrant (because cocofibrant means fibrant) and the replacement of $M_\bullet$ by $I_\bullet$ is called fibrant replacement. (Remark for readers: there is a point here to be cautious. If you want projective resolution to be cofibrant replacement you need to be working with bounded below chain complexes, otherwise you only get DG projectives = cofibrants. Also, you can't have both cof rep = proj res and fib rep = inj res. There are two *different* model structures which provide these examples. See Hovey's book)

Secretly, the reason you can do homotopy theory here is the Dold-Kan correspondence. Studying connective chain complexes is like studying linearized topology. Taking W to be the chain homotopy equivalences gives the category Ho(Ch(R)), a.k.a. $K(R)$. Taking W to be the quasi-isomorphisms gives $\mathcal{D}(R)$. There is a localization from $K(R) \to \mathcal{D}(R)$.

Quillen's brilliant idea was to focus on just these three types of maps, pick out their most important properties, and use these properties to make a definition. A **Model Category** is a category $\mathcal{M}$ with distinguished classes $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ satisfying those properties. The localization described above for spaces works on any model category, i.e. you get a concrete way to make a universal functor $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ taking $\mathcal{W}$ to isomorphisms. So functors $\mathcal{M} \to \mathcal{C}$ which do this induce functors $\operatorname{Ho}\mathcal{M} \to \mathcal{C}$

Model categories are a general place you can do homotopy theory, and this transforms algebraic topology from the study of topological spaces into a general tool useful in many areas of mathematics. This viewpoint let's you do homotopy theory in algebraic geometry, e.g. on the category of Schemes. Voevodsky won a Fields Medal in 2002 by creating the **motivic stable homotopy category** from a model category structure on an enlargement of **Schemes** to resolve the Milnor Conjecture. It also lets you do homotopy theory in homological algebra as we've seen with Ch(R). Quillen coined the phrase **homotopical algebra** for this type of study and it helped him do new **computations in algebraic K-theory**, for which he won a Fields Medal.

Examples of model categories:

- Spaces and Spectra, with homotopy categories HoTop and the stable homotopy category.
- Motivic spaces, where schemes live
- $Ch(R)$ is also a model category with homotopy category = the **derived category** $\mathcal{D}(R)$, which is studied in algebraic geometry and elsewhere. Proving it's triangulated uses the model category structure, which was a hard problem for unbounded chain complexes. Given F, the model category structure helps you **construct from an induced functor between derived categories**, e.g. the left derived functor of an abelianization functor gives **Quillen homology**.
- Chain complexes of quasi-coherent sheaves over a scheme
- Stable module category, where homotopy kills projectives.
- Graphs. Two finite graphs are homotopy equivalent iff they have the same zeta series (reference Beth Malmskog).
- 9 model structures on Set

4. Localization on Monoidal Model Categories

The localization above always lands in a homotopy category and always takes exactly the zig-zags of weak equivalences to isomorphisms. What if we want to invert some map which is not a weak equivalence? Let C be a set of maps in $\mathcal{M}$. Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\ \operatorname{Ho}(\mathcal{M}) \arrow[r] & \operatorname{Ho}(\mathcal{M})[C^{-1}] \end{tikzcd}$$

We'd like a model category $L_C\mathcal{M}$ which actually sits above $\operatorname{Ho}(\mathcal{M})[C^{-1}]$, i.e. we'd like to know the functor $\mathcal{M} \to \operatorname{Ho}(\mathcal{M})[C^{-1}]$ factors through a model category. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want the maps in C to become isomorphisms in $\mathrm{Ho}(\mathcal{M})[C^{-1}]$ so we need them to be weak equivalences in $L_C\mathcal{M}$. So this category must have a **different model category structure**, where $\mathcal{W}' = \langle C \cup \mathcal{W}\rangle$. We've already seen an example of this, when the bottom arrow is $K(R) \to \mathcal{D}(R)$.

This functor L_C is known to preserve some types of structure and destroy others. My interest is in model categories where we can do algebra, i.e. in which we can study monoids, commutative rings, Lie algebras, etc. So my research is concerned with when L_C preserves this kind of structure. For a long time people assumed that it always preserved commutative structure, and in fact this led to a false claim in the proof of the Kervaire Invariant One problem. Mike Hill came up with a counterexample and was later able to patch the proof. That was the starting point of my thesis.

To do algebra we must assume our category is a **monoidal category**, i.e. it comes equipped with a product bifunctor $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ which is associative (i.e. $\otimes\circ(\otimes\times 1) = \otimes\circ(1\times\otimes)$) and has a unit object S (i.e. $S\otimes E = E = E\otimes S$). Here $=$ means naturally isomorphic.

In a monoidal category, a monoid is E with maps $\mu : E\times E\to E$ and $\eta : S\to E$ satisfying some diagrams which encode the usual monoid laws for associativity and unit:

$$\begin{tikzcd} E\otimes(E\otimes E) \arrow[r, "\cong"] \arrow[d, "1\otimes\mu"] & (E\otimes E)\otimes E \arrow[r, "\mu\otimes 1"] & E\otimes E \arrow[d, "\mu"] \\ E\otimes E \arrow[rr, "\mu"] & & E \end{tikzcd} \qquad \begin{tikzcd} S\otimes E \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & E\otimes E \arrow[d, "\mu"] & E\otimes S \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & E & \end{tikzcd}$$

A **commutative monoid** is one which also has a twist isomorphism $\tau : E\otimes E\to E\otimes E$ and more diagrams which say the twist commutes with multiplication.

If $\mathcal{M}$ is a model category and a monoidal category (coherently), then we can ask whether or not $L_C(E)$ is also a commutative monoid. In my thesis I prove a general theorem for when L_C preserves any type of algebraic structure encoded by operads, which are a way to do universal algebra.

Theorem 3. *If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures in the usual way then L_C preserves P-algebras.*

The standard hypotheses for this situation are that $\mathcal{M}$ has the pushout product axiom and the monoid axiom. Without those you have no hope of a good homotopy theory for P-algebras. I also assume other technical but non-restrictive hypotheses on $\mathcal{M}$, and then **I provide conditions on C so that the pushout product and monoid axioms are preserved** (indeed, I classify localization functors which accomplish this; my hypothesis is implied by $L(X\otimes Y)\simeq LX\otimes LY$ or by $L\Sigma\simeq\Sigma L$). For P cofibrant (in the model structure on operads) this is all you need, so I have a very general theorem which states that P-algebra structure is preserved for cofibrant operads. This has led to work recently, joint with Javier Gutiérrez, about when an equivariant operad is cofibrant and when a localization is monoidal.

Cofibrant operads can be thought of as containing the data of algebraic structures which factor in all higher homotopies. A good example is the A_∞ operad; A_∞ algebras are the homotopy theoretic version of associative algebras (note that I do NOT mean they are only associative up to homotopy). Keller gives a nice motivation for these which an algebraist should believe. Given a complex M of A-modules, H^*M is an A_∞-module over A (viewed as an A_∞-algebra) and this structure contains the necessary data to reconstruct M from H^*M. Another example is the following. Suppose $M = \bigoplus_{i=1}^n M_i$ for $M_i\in B$-mod. Suppose we know the extension algebra $Ext_B^*(M,M)$ (this algebra keeps track of short exact sequences) together with its idempotents coming from the M_i.

Can we reconstruct the closure of the M_i under extensions from this? The answer is that the extension algebra is an A_∞ algebra and that structure allows you solve this problem.

The case of commutative monoids is harder, because the operad Com is not cofibrant. So I needed a hypothesis on a model category which guarantees that commutative monoids inherit a model structure. I find a very general condition and it recovers the examples of interest (CDGA over characteristic 0, positive symmetric spectra, simplicial sets). I also find conditions on C to preserve this (one needs $Sym(-)$ to preserve L-equivalences) and in doing so recover the theorem of Hill and Hopkins which patched the Kervaire proof.